Math 3406 HW9

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Problem 1. Assume that A is a matrix that can be diagonalized. Show that

$$\det(e^{At}) = e^{\mathrm{Tr}(A)t}$$

Since A is diagonalizable, it can be expressed as $A = PDP^{-1}$, where P is an invertible matrix and D is a diagonal matrix whose diagonal entries are the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A. The exponential of A is given by:

$$e^{At} = Pe^{Dt}P^{-1}.$$

The expression of the exponential for the diagonalizable matrix:

$$e^A = \sum_{n=0}^{\infty} \frac{(PDP^{-1})^n}{n!} = \sum_{n=0}^{\infty} \frac{PD^nP^{-1}}{n!} = P\left(\sum_{n=0}^{\infty} \frac{D^n}{n!}\right)P^{-1}$$

The matrix e^{Dt} is diagonal with entries $e^{\lambda_i t}$, thus:

$$\det(e^{At}) = \det(Pe^{Dt}P^{-1}) = \det(P)\det(e^{Dt})\det(P^{-1}) = \det(e^{Dt}) = e^{\lambda_1 t+\lambda_2 t+\cdots+\lambda_n t}.$$

The trace of A is the sum of its eigenvalues:

$$\mathrm{Tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n.$$

Thus, we have:

$$\det(e^{At}) = e^{\mathrm{Tr}(A)t}.$$

Problem 2. Let

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Find the eigenvalues and eigenvectors of the matrix $2I - P - P^3$.

$$2I-P-P^3 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2-\lambda & -1 & 0 & -1 \\ -1 & 2-\lambda & -1 & 0 \\ 0 & -1 & 2-\lambda & -1 \\ -1 & 0 & -1 & 2-\lambda \end{bmatrix}.$$

The determinant of the obtained matrix is $\lambda(\lambda-4)(\lambda-2)^2$. Solve the equation $\lambda(\lambda-4)(\lambda-2)^2 = 0$.The roots are: $\lambda_1 = 4, \lambda_2 = 2, \lambda_3 = 2, \lambda_4 = 0$. Next, find the eigenvectors for each eigenvalue:
For $\lambda = 4$:

$$\begin{bmatrix} 2-\lambda & -1 & 0 & -1 \\ -1 & 2-\lambda & -1 & 0 \\ 0 & -1 & 2-\lambda & -1 \\ -1 & 0 & -1 & 2-\lambda \end{bmatrix} = \begin{bmatrix} -2 & -1 & 0 & -1 \\ -1 & -2 & -1 & 0 \\ 0 & -1 & -2 & -1 \\ -1 & 0 & -1 & -2 \end{bmatrix},$$

The eigenvector of this matrix is $\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$.

For $\lambda = 2$:

$$\begin{bmatrix} 2-\lambda & -1 & 0 & -1 \\ -1 & 2-\lambda & -1 & 0 \\ 0 & -1 & 2-\lambda & -1 \\ -1 & 0 & -1 & 2-\lambda \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \end{bmatrix},$$

The eigenvectors of this matrix is $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$.

For $\lambda = 0$:

$$\begin{bmatrix} 2-\lambda & -1 & 0 & -1 \\ -1 & 2-\lambda & -1 & 0 \\ 0 & -1 & 2-\lambda & -1 \\ -1 & 0 & -1 & 2-\lambda \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix},$$

The eigenvector of this matrix is $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

Problem 3. Solve the system

$$\frac{d\vec{u}}{dt} = A\vec{u}, \quad \vec{u}(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

where

$$A = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix}$$

Compute $\lim_{t\to\infty} \vec{u}(t)$.

We first find the eigenvalue and eigenvector of the matrix A, we use the trick to compute 2×2 matrix, and get:

$$-5 : \begin{bmatrix} -1 \\ 1 \end{bmatrix} \to e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$0 : \begin{bmatrix} 3 \\ 2 \end{bmatrix} \to e^{0t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\vec{u}(t) = c_1 e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{0t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \text{where } c_1, c_2 \text{ are constants.}$$

Applying the initial condition $\vec{u}(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, we have:

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$c_1 = -1 \text{ and } c_2 = 1$$

To compute $\lim_{t\to\infty} \vec{u}(t)$, we observe the terms involving e^{-5t} and e^{0t}. Since e^{-5t} approaches 0 as t goes to infinity, the term $c_1 e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ will vanish, leaving us with:

$$\lim_{t\to\infty} \vec{u}(t) = \lim_{t\to\infty} \left(c_1 e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{0t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Problem 4. For the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, find e^{At} using the power series.

The matrix A has two eigenvalues $\lambda_1 = +1$ and $\lambda_2 = -1$ (corresponding to exponentially growing and decaying solutions to $\frac{d\vec{x}}{dt} = A\vec{x}$, respectively). The corresponding eigenvectors are:

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Hence, the matrix exponential should be:

$$\begin{aligned} e^{At} &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} e^t + e^{-t} & e^t - e^{-t} \\ e^t - e^{-t} & e^t + e^{-t} \end{bmatrix}. \end{aligned}$$

Problem 5. Generally, $e^A e^B \neq e^{A+B}$. Check this for the given matrices:

$$A = \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -4 \\ 0 & 0 \end{bmatrix}, \quad A + B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Calculate e^A, e^B, and e^{A+B} and verify whether $e^A e^B = e^{A+B}$.

$$e^A = \begin{bmatrix} e & -4 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e & 4(-1+e) \\ 0 & 1 \end{bmatrix}$$

$$e^B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$$

$$e^{A+B} = \begin{bmatrix} e & 0 \\ 0 & 1 \end{bmatrix}$$

$$e^A e^B = \begin{bmatrix} e & -4+4e \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (e \cdot 1) + (-4+4e) \cdot 0 & (e \cdot -4) + (-4+4e) \cdot 1 \\ (0 \cdot 1) + (1 \cdot 0) & (0 \cdot -4) + (1 \cdot 1) \end{bmatrix} = \begin{bmatrix} e & -4 \\ 0 & 1 \end{bmatrix}$$

As shown, $e^A e^B \neq e^{A+B}$.

Problem 6. A particular solution to the differential equation

$$\frac{d\vec{u}}{dt} = A\vec{u} - \vec{b},$$

where $\vec{b}$ is constant, is given by

$$\vec{u}_p = A^{-1}\vec{b}$$

if A is invertible. The complete solution is $\vec{u}_p + \vec{u}_n$ where $\vec{u}_n$ are solutions to $\frac{d\vec{u}}{dt} = A\vec{u}$.

Find the complete solution for the equations:

a)

$$\frac{du}{dt} = u - 4,$$

The general solution $\mathbf{u}_n$ to the homogeneous equation $\frac{du}{dt} = u$ is:

$$u_n = Ce^t$$

where C is a constant.

A particular solution $\mathbf{u}_p$ can be found by assuming $u_p = k$ where k is a constant. Substituting u_p into the equation gives:

$$0 = k - 4 \implies k = 4.$$

Thus, the complete solution is:

$$u(t) = u_p + u_n = 4 + Ce^t.$$

b)

$$\frac{d\vec{u}}{dt} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \vec{u} - \begin{bmatrix} 4 \\ 6 \end{bmatrix}.$$

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Multiplying A^{-1} by $\vec{b}$ gives the particular solution $\vec{u}_p$:

$$\vec{u}_p = A^{-1}\vec{b} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

The homogeneous solution $\vec{u}_n$ solves $\frac{d\vec{u}}{dt} = A\vec{u}$. The eigenvalues of A are both 1, thus $\vec{u}_n$ can be written as:

$$\vec{u}_n = C_1 e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + C_2 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where C_1 and C_2 are constants.

Hence, the complete solution is:

$$\vec{u}(t) = \vec{u}_p + \vec{u}_n = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + C_1 e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + C_2 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Problem 7. The Pauli matrices are given by

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

a) Show that every self-adjoint matrix $H = H^*$ can be written as

$$H = aI_2 + b\sigma_1 + c\sigma_2 + d\sigma_3$$

where a, b, c, d are real numbers.

First note that any self-adjoint matrix can be written in terms of the Pauli matrices and the identity matrix.

Let H be a self-adjoint matrix. Then, by definition, $H^* = H$. We can write H in terms of the Pauli matrices and the identity matrix as follows:

$$H = aI_2 + b\sigma_1 + c\sigma_2 + d\sigma_3$$

where a, b, c, d are complex numbers.

To show that a, b, c, d must be real, we consider the condition for H to be self-adjoint:

$$H^* = H$$

For H to be self-adjoint, we need:

$$H = H^*$$

$$aI_2 + b\sigma_1 + c\sigma_2 + d\sigma_3 = a^* I_2 + b^* \sigma_1 + c^* \sigma_2 + d^* \sigma_3$$

Since a, b, c, d are equal to their complex conjugates, they must be real. Therefore, we can write H as

$$H = aI + b\sigma_1 + c\sigma_2 + d\sigma_3$$

where a, b, c, d are real numbers.

b) Compute the eigenvalues of $b\sigma_1 + c\sigma_2 + d\sigma_3$ and use this to compute the eigenvalues of H. (Hint: Show that $(b\sigma_1 + c\sigma_2 + d\sigma_3)^2 = (b^2 + c^2 + d^2)I_2$ and note that the trace of $b\sigma_1 + c\sigma_2 + d\sigma_3$ is zero.)

Let $B = b\sigma_1 + c\sigma_2 + d\sigma_3$. We compute:

$$B^2 = (b\sigma_1 + c\sigma_2 + d\sigma_3)^2 = b^2\sigma_1^2 + c^2\sigma_2^2 + d^2\sigma_3^2 + \text{(cross terms)}.$$

Using properties of Pauli matrices:

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I_2,$$
$$\sigma_i\sigma_j + \sigma_j\sigma_i = 0 \quad \text{for } i \neq j.$$

Thus, $B^2 = (b^2 + c^2 + d^2)I_2$.

Since $B^2 = (b^2 + c^2 + d^2)I_2$, the eigenvalues of B are $\pm\sqrt{b^2 + c^2 + d^2}$.

Considering $H = aI_2 + B$, the eigenvalues of H are:

$$a + \sqrt{b^2 + c^2 + d^2} \quad \text{and} \quad a - \sqrt{b^2 + c^2 + d^2}.$$

Problem 8.

a) Let T be an upper triangular $n \times n$ matrix with diagonal elements $a_1, a_2, \ldots, a_n$. Show that

$$(T - a_1 I)(T - a_2 I) \cdots (T - a_n I) = \text{zero matrix}$$

In other words, the matrix T is a 'root' of the characteristic polynomial.

Given an upper triangular matrix T, consider the subspace W spanned by the standard basis vectors $\epsilon_1, \epsilon_2, \ldots, \epsilon_{n-1}$. This subspace is invariant under the action of T, which means that T maps vectors in W to other vectors in W. The matrix B, which is the representation of T restricted to W, is upper triangular and is obtained by deleting the last row and column of T.

The characteristic polynomial of B is $g(x) = (x - a_1)(x - a_2) \cdots (x - a_{n-1})$. By the inductive hypothesis, we assume that $g(B) = 0$ for matrices of size less than n, which implies $g(B)\epsilon_i = 0$ for all $i \leq n - 1$.

Considering the characteristic polynomial $f(x) = (x - a_1)(x - a_2) \cdots (x - a_n)$ of the full matrix T, we have $f(T) = (T - a_n I)g(T)$. It follows that $f(T)\epsilon_i = 0$ for all $i \leq n-1$, given that $g(T)\epsilon_i = 0$ for these indices.

To show that $f(T)\epsilon_n = 0$, we note that $(T - a_n I)\epsilon_n$ is a linear combination of $\epsilon_1, \ldots, \epsilon_{n-1}$ due to the upper triangular structure of T. Thus:

$$f(T)\epsilon_n = g(T)(T - a_n I)\epsilon_n = g(T)\left(r_1\epsilon_1 + \ldots + r_{n-1}\epsilon_{n-1}\right) = r_1 g(T)\epsilon_1 + \ldots + r_{n-1} g(T)\epsilon_{n-1} = 0.$$

As $f(T)\epsilon_i = 0$ for all basis vectors ϵ_i, where $i = 1, \ldots, n$, and since the action of a linear operator on the basis vectors determines the operator entirely, it follows that $f(T) = 0$, the zero matrix.

b) Using the Schur factorization $A = UTU^*$ where U is unitary and T upper triangular, prove Cayley's theorem which states that the matrix is a root of its characteristic polynomial. More precisely, if the characteristic polynomial of A is given by

$$(-1)^n \lambda^n + (-1)^{n-1} \mathrm{Tr}(A) \lambda^{n-1} + \cdots + \det(A)$$

then

$$(-1)^n A^n + (-1)^{n-1} \mathrm{Tr}(A) A^{n-1} + \cdots + (\det(A)) I_n = 0$$

From the Schur factorization $A = UTU^*$, where T has the eigenvalues of A on its diagonal, we know that T satisfies its characteristic polynomial. Thus, we have

$$(T - \lambda_1 I)(T - \lambda_2 I) \cdots (T - \lambda_n I) = 0,$$

where λ_i are the eigenvalues of A.

The characteristic polynomial $p(\lambda)$ of A can be written as:

$$p(\lambda) = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n).$$

Evaluating this polynomial at A gives:

$$p(A) = (-1)^n (A - \lambda_1 I)(A - \lambda_2 I) \cdots (A - \lambda_n I).$$

Substituting A with UTU^* and considering $UU^* = I$, we get:

$$p(A) = (-1)^n (UTU^* - \lambda_1 I)(UTU^* - \lambda_2 I) \cdots (UTU^* - \lambda_n I).$$

By multiplying out these terms and using the fact that T is a root of its characteristic polynomial, we obtain:

$$p(A) = (-1)^n U(T - \lambda_1 I)(T - \lambda_2 I) \cdots (T - \lambda_n I)U^* = 0.$$

Thus, A also satisfies its characteristic polynomial.