Math 3406 HW7

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Problem 1. If A^+ is the pseudo inverse of A, using the formulas for A^+, show that $(A^+A)^2 = A^+A$ and $(AA^+)^2 = (AA^+)$. Also verify that $(A^+A)^T = (A^+A)$ and $(AA^+)^T = (AA^+)$. This is an explicit verification that AA^+ and A^+A are projections.

From class, we have

$$A^+A = R^+C^+CR = R^+R = \text{Projection onto row space of A}$$

$$AA^+ = CRR^+C^+ = CC^+ = \text{Projection onto column space of A}$$

$$RR^+ = I \quad C^+C = I$$

$$A^+ = (A^TA)^{-1}A^T$$

1. Show $(A^+A)^2 = A^+A$

$$\begin{aligned}(A^+A)^2 &= (A^+A)(A^+A)\\ &= R^+(RR^+)R\\ &= R^+IR\\ &= R^+R\\ &= A^+A\end{aligned}$$

   Conceptually, $(A^+A)^2$ is the projection matrix onto the row space of (A^+A) which is equivalent to (A^+A).

2. Show $(AA^+)^2 = AA^+$

$$\begin{aligned}(AA^+)^2 &= (AA^+)(AA^+)\\ &= CC^+CC^+\\ &= CIC^+\\ &= CC^+\\ &= AA^+\end{aligned}$$

   Conceptually, $(AA^+)^2$ is the projection matrix onto the column space of AA^+ which is equivalent to AA^+.

3. Verify $(A^+A)^T = A^+A$

$$\begin{aligned}
(A^+A)^T &= ((A^TA)^{-1}A^TA)^T \\
&= I^T \\
&= I \\
&= (A^TA)^{-1}A^TA \\
&= A^+A
\end{aligned}$$

4. Verify $(AA^+)^T = AA^+$

$$\begin{aligned}
(AA^+)^T &= (A^+)^TA^T \\
&= A((A^TA)^{-1})^TA^T \\
&= A((A^TA)^T)^{-1}A^T \\
&= A(A^TA)^{-1}A^T \\
&= AA^+
\end{aligned}$$

Problem 2. Compute A^{++}, i.e., the pseudo-inverse of the pseudo-inverse of A.

From class:

1. $A^+ = R^+C^+$
2. $R^+ = R^T(RR^T)^{-1}$
3. $C^+ = (C^TC)^{-1}C^T$

We start by applying R^+ and C^+ to compute A^+ and then A^{++}

$$\begin{aligned}
A^+ &= R^+C^+ \\
A^{++} &= (R^+C^+)^+ \\
&= (C^+)^+(R^+)^+ \\
&= ((C^TC)^{-1}C^T)^+(R^T(RR^T)^{-1})^+ \\
&= (C^TC)^{-1}C^TCRR^TR(RR^T)^{-1} \\
&= (C^TC)^{-1}(C^TC)CRR^T(RR^T)^{-1}(RR^T) \\
&= ICRI \\
&= CR \\
&= A
\end{aligned}$$

Problem 3. Let A be a 2×2 matrix. Consider the following statements and determine if they are true or false, providing a reason if true or a counterexample if false:

1. The determinant of $I + A$ is $1 + \det(A)$.

   False. The determinant of $I+A$ is not necessarily $1+\det(A)$. For example, let
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$
   Then
$$I + A = \begin{pmatrix} 1+a & b \\ c & 1+d \end{pmatrix},$$
   and
$$\det(I + A) = (1 + a)(1 + d) - bc.$$
   This is not necessarily equal to $1 + (ad - bc) = 1 + \det(A)$.

2. The determinant of $4A$ is $4 \det(A)$.

   False. The determinant of $4A$ is $16 \det(A)$. For a 2×2 matrix A, when the matrix is multiplied by a scalar k, the determinant is multiplied by k^2 because the matrix has two rows (and columns).

3. The determinant of $AB - BA$ is always zero.

   False. Consider the following example where $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $AB - BA$ is computed as
$$AB = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix},$$
   which yields
$$AB - BA = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} - \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} 0 & -b \\ c & 0 \end{bmatrix}$$
   with determinant bc. This is not zero unless $b = 0$ or $c = 0$.

4. If the entries in every row of a matrix A add to zero show that $\det A = 0$.

If the entries of every row of a matrix A add to zero, then $A\vec{x} = \vec{0}$ when $\vec{x} = (1, \ldots, 1)^T$, since each component of $A\vec{x}$ is the sum of the entries in a row of A. Since A has a non-zero nullspace, it is not invertible and $\det(A) = 0$.

5. Use row operations to compute the determinant of the matrix

$$\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$$

Using row operations and properties of the determinant, we have:

$$\begin{aligned}
\det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} &= \det \begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 1 & c & c^2 \end{bmatrix} \\
&= (b-a) \det \begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{bmatrix} \\
&= (b-a) \det \begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 1 & c-a & (c-a)(c-b) \end{bmatrix} \\
&= (b-a)(c-a)(c-b) \det \begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & 1 \end{bmatrix} \\
&= (b-a)(c-a)(c-b) \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= (b-a)(c-a)(c-b).
\end{aligned}$$

Compute the determinant of

$$\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{bmatrix}$$

P is a polynomial with degree 3, and x_2, x_3, x_4 are its roots, so

$$P(x) = \lambda(x - x_2)(x - x_3)(x - x_4).$$

Thus, the given determinant is

$$P(x_1) = \lambda(x_1 - x_2)(x_1 - x_3)(x_1 - x_4),$$

and finally, to figure out the leading coefficient λ, we have

$$\lambda = \det \begin{pmatrix} 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ 1 & x_4 & x_4^2 \end{pmatrix} := V_3.$$

So induction, we have

$$\lambda = (x_2 - x_3)(x_2 - x_4)(x_3 - x_4).$$

which, when expanded, becomes

$$\det = (x_2 - x_1)(x_3 - x_1)(x_4 - x_1)(x_3 - x_2)(x_4 - x_2)(x_4 - x_3)$$

This represents the product of the differences between each pair of x variables, ensuring that each pair is considered exactly once

6. The Big Formula for the determinant has 24 terms if A is 4×4. How many terms include a_{13} and a_{22}? Try to reason without writing down all 24 terms, which could be a challenge.

We first write out the expression for this matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

The term that involves a_{13} and a_{22}, we can isolate one terms from the general formula for determinant as:

$$a_{13}a_{22} \det \begin{bmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{bmatrix} = a_{13}a_{22}(a_{31}a_{44} - a_{34}a_{41})$$

So there are in total 2 terms that include a_{13} and a_{22}.

The explanation as follows:

We select a_{13} and a_{22}, and we are left with two rows (the third and the fourth) and two columns (the first and the fourth) to fill the remaining two positions in the term. There are 2! ways to do this (either by choosing the entry from the third row and first column and the entry from the fourth row and fourth column, or vice versa).Thus, without writing down all 24 terms, we can reason that there are 2 terms in the determinant of A that include both a_{13} and a_{22}.

7. If the edge vectors of a parallelogram in $\mathbb{R}^2$ have lengths $\|a_1\| = 1$ and $\|a_2\| = 2$, what is the largest area the parallelogram could have?

From class, we talked about the area of the square transformed by S into parallelogram as the determinant of an 2×2 matrix. And the area between two vectors are the largest when two vectors are orthogonal to each other, so we can construct such matrix as:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

and the area is the determinant which is 2.

Another way to explain this is using cross-product which represent the area of the parallelogram spanned by two vector,

$$a_1 \times a_2 = \|a_1\|\|a_2\| \sin(\theta)$$

To find the largest area, we let $\sin(\theta) = 1$, and thus area equals 2.

8. (Hadamard's Inequality) Prove that for any $n\times n$ matrix $A = [\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n]$, the following inequality holds:

$$|\det(A)| \leq \|\vec{a}_1\|\|\vec{a}_2\| \cdots \|\vec{a}_n\|.$$

Hint: Consider the factorization $A = QR$ and show that the diagonal elements $|R_{ii}| \leq \|\vec{a}_i\|$.

Proof. The determinant of A is given by:

$$\det(A) = \det(QR) = \det(Q)\det(R)$$

Since Q is orthogonal, $\det(Q) = \pm 1$, and therefore:

$$\det(A) = \pm \det(R)$$

The determinant of an upper triangular matrix, like R, is the product of its diagonal elements:

$$\det(R) = R_{11} \cdot R_{22} \cdot \ldots \cdot R_{nn}$$

The norm of each column vector $\vec{a}_i$ of A is at least as large as the absolute value of the corresponding diagonal element R_{ii} because each $\vec{a}_i$ can be expressed as a linear combination of the orthonormal columns of Q with coefficients from R. Hence, $|R_{ii}| \leq \|\vec{a}_i\|$ for all i.

Combining these facts, we find:

$$|\det(A)| = |\det(R)| = |R_{11} \cdot R_{22} \cdot \ldots \cdot R_{nn}| \leq \|\vec{a}_1\|\|\vec{a}_2\| \cdots \|\vec{a}_n\|$$

□