Math 3406 HW6

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Problem 1. Find the orthogonal vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ using Gram-Schmidt from a, b, c where

$$\vec{a} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{c} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

We apply Gram-Schmidt to a, b, c. We have

$$\vec{A} = \frac{a}{\|a\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

Next,

$$\vec{B} = \frac{b - (b \cdot A)\, A}{\|b - (b \cdot A)\, A\|} = \frac{\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \left(-\frac{\sqrt{2}}{2}\right) \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}}{\left\| \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix} \right\|} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{\sqrt{2}}{3} \\ 0 \end{bmatrix}.$$

Finally,

$$\vec{C} = \frac{c - (c \cdot A)\, A - (c \cdot B)\, B}{\|c - (c \cdot A)\, A - (c \cdot B)\, B\|} = \frac{\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} - \left(\frac{1}{2\sqrt{2}}\right) \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} - \left(\frac{1}{3}\right) \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}}{\left\| \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix} \right\|} = \begin{bmatrix} \frac{1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} \\ -\frac{\sqrt{3}}{2} \end{bmatrix}.$$

Problem 2.

a) Find orthonormal vectors $\vec{q}_1, \vec{q}_2, \vec{q}_3$ such that $\vec{q}_1, \vec{q}_2$ span the column space of the matrix

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}.$$

Let $\vec{c}_1$ and $\vec{c}_2$ be the columns of A. Then

$$\vec{q}_1 = \frac{\vec{c}_1}{\|\vec{c}_1\|} = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

$$\vec{q}_2 = \frac{\vec{c}_2 - \vec{q}_1 \cdot (\vec{c}_2^T \vec{q}_1)}{\|\vec{c}_2 - \vec{q}_1 \cdot (\vec{c}_2^T \vec{q}_1)\|} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

For $\vec{q}_3$, pick an arbitrary vector $\vec{c}_3$ that is not in the column space of A, and continue doing Gram-Schmidt. For example, we can pick $\vec{c}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ because after row reduced, we found the third row is empty. Then compute

$$\vec{q}_3 = \frac{\vec{c}_3 - \vec{q}_1 \cdot (\vec{q}_1^T \vec{c}_3) - \vec{q}_2 \cdot (\vec{q}_2^T \vec{c}_3)}{\|\vec{c}_3 - \vec{q}_1 \cdot (\vec{q}_1^T \vec{c}_3) - \vec{q}_2 \cdot (\vec{q}_2^T \vec{c}_3)\|} = \frac{1}{3}\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

b) Which of the four fundamental subspaces contains $\vec{q}_3$?
$\vec{q}_3 \in N(A^T)$, since $N(A^T) \perp C(A)$, and $\vec{q}_3 \perp C(A)$.

c) Solve $A\vec{x} = \vec{b}$ by least squares where

$$\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}.$$

$$A^T A\vec{x} = A^T \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 9 & -9 \\ -9 & 18 \end{bmatrix} \vec{x} = \begin{bmatrix} -9 \\ 27 \end{bmatrix}$$

$$\vec{x} = \left(\begin{bmatrix} 9 & -9 \\ -9 & 18 \end{bmatrix}\right)^{-1} \begin{bmatrix} -9 \\ 27 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -9 \\ 27 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Problem 3. Find all matrices that are orthogonal and lower triangular.

By the definition of orthogonality that $A^{-1} = A^T$. Note that the inverse of a lower triangular matrix must also be lower triangular. Thus, A^T is both lower and upper (as it is the transpose of a lower triangular matrix); we conclude that A^T, and hence A, is diagonal. Let d_i be the ith diagonal entry of A. Then, $A^T = A$, so the ith diagonal entry of $A^T A$ is d_i^2. As $A^T A$ also equals the identity matrix, we have $d_i^2 = 1 \Rightarrow d_i = \pm 1$.

Problem 4. Assuming $ad - bc > 0$, the QR factorization of the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Given Q should be orthogonal matrix, we have $Q^T Q = I$. We can pick any column in R^2 that the length is 1, so we can pick the first column vector to be the unit circle: $\vec{v_1} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$. To choose the second column vector that is orthogonal to $\vec{v_1}$, we can pick $\vec{v_2} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$. Since $ad - bc > 0$, the matrix A is invertible. To construct a general representation of $A = QR$, we can set Q the unique orthogonal matrix with det $= 1$. Since R means upper triangular , we can set two columns as $\vec{w_1} = \begin{bmatrix} x \\ 0 \end{bmatrix}$, and $\vec{w_2} = \begin{bmatrix} z \\ y \end{bmatrix}$, as follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x & z \\ 0 & y \end{bmatrix}$$

Now, given Q is orthogonal, we have the following properties:

$$\langle Qv, Qw \rangle = \langle v, w \rangle$$

$$||v_1||^2 = ||w_1||^2$$

$$||v_2||^2 = ||w_2||^2$$

$$\langle v_1, v_2 \rangle = \langle w_1, w_2 \rangle$$

Following these properties, we have the following equation:

$$x^2 = a^2 + c^2 \Rightarrow x = \sqrt{a^2 + c^2}$$

$$xz = ab + cd \Rightarrow z = \frac{ab + cd}{\sqrt{a^2 + c^2}}$$

$$y^2 + z^2 = b^2 + d^2 \Rightarrow y = \frac{ad - bc}{\sqrt{a^2 + c^2}}$$

Now, we apply trigonometry, we get:

$$\cos\theta = \frac{a}{\sqrt{a^2 + c^2}} \qquad \sin\theta = \frac{c}{\sqrt{a^2 + c^2}}$$

At last, we plug in back to the original construction:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{a}{\sqrt{a^2+c^2}} & -\frac{c}{\sqrt{a^2+c^2}} \\ \frac{c}{\sqrt{a^2+c^2}} & \frac{a}{\sqrt{a^2+c^2}} \end{bmatrix} \begin{bmatrix} \sqrt{a^2 + c^2} & \frac{ab+cd}{\sqrt{a^2+c^2}} \\ 0 & \frac{ad-bc}{\sqrt{a^2+c^2}} \end{bmatrix}$$

Problem 5. Find the QR factorization of the matrix

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}$$

To find Q, we first compute Gram-Schmidt. To find Q, we first compute Gram-Schmidt.

$$\vec{q_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{a}_2 = \vec{u}_2 - (\vec{u}_2 \cdot \vec{q}_1)\vec{q}_1 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \vec{q_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{a}_3 = \vec{u}_3 - (\vec{u}_3 \cdot \vec{q}_1)\vec{q}_1 - (\vec{u}_3 \cdot \vec{q}_2)\vec{q}_2 = \vec{u}_3 - 2\vec{q}_1 = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}, \vec{q_3} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

And thus we get Q:

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, Q^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

, We move on to find R:

$$R = Q^T A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

Finally, we check:

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

Problem 6. Consider the system of equations $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 6 \\ 6 \\ 3 \end{bmatrix}.$$

Find the solution of least length. Do this in two ways, using calculus and using the fact that the solution of least length must be perpendicular to the null space of A.

Approach 1: Calculus

We find the general solution for the equation $A\vec{x} = \vec{b}$:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 3 & 2 & 1 & 6 \\ 1 & 1 & 1 & 3 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The general solution is:

$$y + 2z = 3, \quad x - z = 0, \quad z \text{ is free}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} z \\ 3 - 2z \\ z \end{bmatrix}$$

To find the least length, we take the length of the vector, and then take derivative and set it to 0:

$$||x, y, z|| = z^2 + (3 - 2z)^2 + z^2$$

$$\frac{\partial f}{\partial z} = 4z - 4(3 - 2z) = 0 \Rightarrow z = 1$$

At last, we plug in back and get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ 3 - 2z \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Approach 2: the fact that the solution of least length must be perpendicular to the null space of A.

We already done the row reduced form, and we know that the dimension of the row space is 2, and so we pick two vectors from row space to write as the solution x, and we pick:

$$x = \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$Ax = \alpha \begin{bmatrix} 6 \\ 6 \\ 3 \end{bmatrix} + \beta \begin{bmatrix} 14 \\ 10 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 3 \end{bmatrix}$$

So, $\alpha = 1, \beta = 0$, and the solution to least length $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Problem 7. Find the pseudo inverse of the matrices A and B, where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 4 \\ 0 & 0 \end{bmatrix}$$

We take the formula derived from class:

$$A^+ = R^+Q^T = R^T(RR^T)^{-1}Q^T$$

$$A = QR = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\sqrt{2}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

We plug in Q and R in the equation. After the computation, we get

$$A^+ = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Similar for B, since B is already in row reduced form, we write B as

$$B = CR = \begin{bmatrix} 2 & 0 \\ 0 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We take the formula derived from class:

$$B^+ = R^+C^+ = R^T(RR^T)^{-1}(C^TC)^{-1}C^T$$

We plug in c and R in the equation. After the computation, we get

$$B^+ = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 \end{bmatrix}$$

Problem 8. For the matrix $A = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and the matrix $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, show that the pseudo-inverse of AB, $(AB)^+$, is not B^+A^+ but $(BA)^+ = A^+B^+$.

It is easy to notice that the dimension of $(AB)^+$ does not match the dimension of B^+A^+, as noticed that (AB) is an 1×1 matrix, and (BA) is an 2×2 matrix. We can just do direct computation to verify as follows:

$$AB = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \quad BA = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$A^+ = (A^TA)^{-1}A^T.$$

$$A^+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad B^+ = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$(AB)^+ = 1 \neq \frac{1}{2} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = B^+ A^+$$

$$(BA)^+ = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} = A^+ B^+$$