Math 3406 HW5

Pengfei Zhu

February 23, 2024

Problem 1. Suppose A is the 4×4 identity matrix with its last column removed, i.e., A is a 4×3 matrix. Project the vector $\vec{b} = [1, 2, 3, 4]^T$ onto the column space of A. What is the orthogonal projection that projects $\mathbb{R}^4$ onto the column space of A?

We have two similar approaches to this problem:

1. Find the orthogonal projection P using the direct computation $P = A(A^T A)^{-1} A^T$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, A^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and so we have:

$$p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here we found the orthogonal projection matrix P that lose b_4 after the projection. As a result, the projection of $\vec{b}$ is $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ 0 \end{bmatrix}$

2. Observe that A is a upper-triangular matrix, we can easily determine the QR factorization of A where Q is just A. And so we can simply compute

$$Rx = Q^T b = A^T b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

we also know that R has to be an identity in $\mathbb{R}^4$, so we take back substitution and get the same result.

Problem 2. To find the projection matrix P onto the plane $x - y - 2z = 0$, choose two vectors in that plane and make them the columns of A. The plane will be the column space of A. Compute $P = A(A^T A)^{-1} A^T$.

We view the plane as the null space of the matrix $\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$. Then a basis of this space is given by $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Hence we take the matrix A to be $A = \begin{bmatrix} -2 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$. Thus,

$$A^T A = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}$$

and

$$(A^T A)^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{5}{6} \end{bmatrix}$$

Then the projection matrix is

$$P = A(A^T A)^{-1} A^T = \begin{bmatrix} -2 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{5}{6} \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} & -\frac{1}{3} \\ \frac{1}{6} & \frac{5}{6} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

.

Problem 3. Write down three equations for the line $C + Dt$ to go through $b = 7$ at $t = -1$, $b = 7$ at $t = 1$, and $b = 21$ at $t = 2$. Find the least square solution $\vec{x}^* = (C, D)$.

The equations for the line $b = C + Dt$ are given by

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}$$

Thus, the least-squares solution is given by solving

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix},$$

which simplifies to

$$\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix}.$$

Hence, $C = 9$, $D = 4$, and then $\vec{x}^* = (9, 4)$.

Problem 4. The two lines $t(1,1,1)$ and $(s,3s,-1)$ do not meet. Find the distance between these two lines.

For the first line $t(1,1,1)$, the direction vector is $\vec{d_1} = (1,1,1)$.

For the second line $(s,3s,-1)$, the direction vector is $\vec{d_2} = (1,3,0)$.

Now, we find a vector $\vec{v}$ that is perpendicular to both direction vectors $\vec{d_1}$ and $\vec{d_2}$ using the cross product:

$$\vec{v} = \vec{d_1} \times \vec{d_2} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$$

Now, let's choose arbitrary points on each line, for example plug in 0 to both equations:

For the first line, let $P_1 = (0,0,0)$. For the second line, let $P_2 = (0,0,-1)$.

Now, we find the vector $\vec{w}$ connecting P_1 and P_2:

$$\vec{w} = P_2 - P_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Finally, we find the distance d between the two lines using the projection of $\vec{w}$ onto $\vec{v}$:

$$d = \frac{|\vec{v} \cdot \vec{w}|}{|\vec{v}|} = \frac{\left| \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right|}{\left| \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} \right|} = \frac{|0+0+2|}{\sqrt{(-3)^2 + 1^2 + 2^2}} = \frac{2}{\sqrt{14}} = \frac{2\sqrt{14}}{14} = \frac{\sqrt{14}}{7}$$

So, the distance between the two lines is $\frac{\sqrt{14}}{7}$.

Another approach is that we consider two line as a matrix A, and express it in the form Ax=b, shown as follows:

$$\begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

and we perform projection on matrix A, which is $P = A(A^TA)^{-1}A^T$,

Now, compute P:

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{5}{7} & -\frac{2}{7} \\ -\frac{2}{7} & \frac{3}{14} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 0 \end{bmatrix} = \begin{bmatrix} \frac{5}{14} & \frac{3}{14} & \frac{3}{7} \\ \frac{3}{14} & \frac{13}{14} & -\frac{1}{7} \\ \frac{3}{7} & -\frac{1}{7} & \frac{5}{7} \end{bmatrix}$$

$$P\vec{x}-\vec{b}=\begin{bmatrix}\frac{5}{14} & \frac{3}{14} & \frac{3}{7}\\ \frac{3}{14} & \frac{13}{14} & -\frac{1}{7}\\ \frac{3}{7} & -\frac{1}{7} & \frac{5}{7}\end{bmatrix}\begin{bmatrix}0\\0\\-1\end{bmatrix}-\begin{bmatrix}0\\0\\-1\end{bmatrix}=\begin{bmatrix}-\frac{3}{7}\\ \frac{1}{7}\\ \frac{2}{7}\end{bmatrix}$$

Now, compute the magnitude of $P\vec{x}-\vec{b}$:

$$\|P\vec{x}-\vec{b}\|=\sqrt{\left(-\frac{3}{7}\right)^2+\left(\frac{1}{7}\right)^2+\left(\frac{2}{7}\right)^2}=\frac{\sqrt{14}}{7}$$

This gives the distance between the two lines.

Problem 5.

a) If a matrix A has three orthogonal columns each of length 4, what is A^TA?

The matrix A looks like $(\vec{q}_1\ \vec{q}_2\ \vec{q}_3)$. Then

$$A^TA=\begin{bmatrix}\vec{q}_1\cdot\vec{q}_1 & \vec{q}_1\cdot\vec{q}_2 & \vec{q}_1\cdot\vec{q}_3\\ \vec{q}_2\cdot\vec{q}_1 & \vec{q}_2\cdot\vec{q}_2 & \vec{q}_2\cdot\vec{q}_3\\ \vec{q}_3\cdot\vec{q}_1 & \vec{q}_3\cdot\vec{q}_2 & \vec{q}_3\cdot\vec{q}_3\end{bmatrix}=\begin{bmatrix}16 & 0 & 0\\ 0 & 16 & 0\\ 0 & 0 & 16\end{bmatrix}=16\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=16I_3$$

b) If A has three orthogonal columns of length 1, 2, 3, what is A^TA?

The matrix A looks like $(\vec{q}_1\ \vec{q}_2\ \vec{q}_3)$. Then

$$A^TA=\begin{bmatrix}\vec{q}_1\cdot\vec{q}_1 & \vec{q}_1\cdot\vec{q}_2 & \vec{q}_1\cdot\vec{q}_3\\ \vec{q}_2\cdot\vec{q}_1 & \vec{q}_2\cdot\vec{q}_2 & \vec{q}_2\cdot\vec{q}_3\\ \vec{q}_3\cdot\vec{q}_1 & \vec{q}_3\cdot\vec{q}_2 & \vec{q}_3\cdot\vec{q}_3\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 9\end{bmatrix}$$

Problem 6. If Q has orthonormal columns, what is the least squares solution $\vec{x}^*$ of $Q\vec{x}=\vec{b}$?

The least squares solution can be written as $\vec{x}^*=(Q^TQ)^{-1}Q^T\vec{b}$. By definition, Q is orthonormal implies $Q^TQ=I$. Therefore, $\vec{x}^*=Q^Tb$.

Alternatively, solving least squares means finding a solution to $Q^TQ\vec{x}^*=Q^T\vec{b}$. Since $Q^TQ=I$, we can conclude $\vec{x}^*=Q^T\vec{b}$.

Problem 7. Suppose P_1 and P_2 are orthogonal projection matrices, i.e., $P_i^2=P_i$, $P_i^\top=P_i$, $i=1,2$. Show that P_1P_2 is an orthogonal projection matrix if and only if $P_1P_2=P_2P_1$.

Proof.

$\Rightarrow$ By definition, a linear operator P is an orthogonal projection if and only if $P^2=P$ and $P^T=P$. If P_1P_2 is an orthogonal projection, then

$$P_1P_2=(P_1P_2)^T=P_2^TP_1^T=P_2P_1,$$

where we use the identity $(AB)^T=B^TA^T$ and the orthogonality of the P_i.

$\Leftarrow$ if $P_1P_2 = P_2P_1$, then

$$P^2 = P_1P_2P_1P_2 = P_1(P_1P_2)P_2 = P_1^2P_2^2 = P_1P_2 = P,$$

$$P^* = (P_1P_2)^T = P_2^TP_1^T = P_2P_1 = P_1P_2 = P,$$

so P_1P_2 is an orthogonal projection. $\square$

Problem 8. An $m \times m$ matrix A has rank m and satisfies $A^2 = A$. What is A? Give a reason for your answer.

Since A is $m \times m$ matrix A with full rank, it means very there are no free variables since the vectors are linearly independent and hence every column has a pivot. As a result, matrix A is invertible. We perform inverse on both side:

$$A^{-1}A^2 = A^{-1}A$$

$$A = I_m$$

Conceptually, since $A^2 = A$, it follows the definition of a projection matrix. In this case, since A has rank m, it projects vectors onto an m-dimensional subspace of the original vector space. Now, if the rank of A is m, it means A projects onto the entire m-dimensional space. In other words, for any vector v in the m-dimensional space, $Av = v$. Thus, the matrix A must be the identity matrix I_m, because it leaves vectors unchanged when multiplied by A. Therefore, $A = I_m$.