Math 3406 HW4

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Problem 1. Give a reason or a counterexample for the following statements:

a) A and A^T have the same number of pivots.
True. The number of pivots of A is its column rank, r. We know that the column rank of A equals the row rank of A, which is the column rank of A^T. Hence, A^T must have the same number of pivots as A.

b) A and A^T have the same left null space.
False.

Counterexample: Take any 1×2 matrix, $A = \begin{bmatrix} a & b \end{bmatrix}$. The left nullspace of A contains vectors in $\mathbb{R}$, while the left nullspace of A^T, which is the right nullspace of A, contains vectors in $\mathbb{R}^2$, so they cannot be the same.

c) If the row space equals the column space then $A = A^T$.
False.

Counterexample: Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Here, the row space and the column space are both equal to all of $\mathbb{R}^2$ (since A is invertible), but $A \neq A^T$.

d) If $A^T = -A$ then the row space of A equals the column space.
True. The row space of A equals the column space of A^T, which for this particular A equals the column space of $-A$. Since A and $-A$ have the same fundamental subspaces, by part (b) of the previous question, we conclude that the row space of A equals the column space of A.

Problem 2. Fredholm alternative: The equation $A\tilde{x} = \tilde{b}$ has a solution if and only if $\tilde{b}$ is perpendicular to $N(A^T)$.

Proof.

$\Rightarrow$ If $Ax = b$ has a solution, then b is perpendicular to $N(A^T)$.

If $Ax = b$ has a solution, it means that b lies in the column space of A. Since any vector in the column space of A is perpendicular to any vector in $N(A^T)$, b must be perpendicular to $N(A^T)$.

$\Leftarrow$ If b **i**s perpendicular to $N(A^T)$, then $Ax = b$ has a solution.

If b is perpendicular to $N(A^T)$, it means that b lies in the column space of A. Consequently, b can be expressed as a linear combination of the columns of A, i.e., $b = Ax$ for some x. Thus, the equation $Ax = b$ has a solution.

Therefore, we have proven both directions of the Fredholm alternative Theorem. $\square$

Problem 3. Suppose that A is 3×4 and B is 4×5 and $AB = 0$. So $N(A)$ contains $C(B)$. Deduce from the dimensions of $N(A)$ and $C(B)$ that $\text{rank}(A) + \text{rank}(B) \leq 4$.

By the fundamental theorem of linear algebra, the dimension of $N(A)$ is

$$N(A) = 4 - \text{rank}(A)$$

The dimension of $C(B)$ is $\text{rank}(B)$ since $C(B)$ is spanned by the columns of B.

Given that $C(B) \subseteq N(A)$, we have:

$$\text{rank}(B) \leq 4 - \text{rank}(A)$$

Rearranging this inequality, we get:

$$\text{rank}(A) + \text{rank}(B) \leq 4$$

Therefore, we have deduced from the dimensions of $N(A)$ and $C(B)$ that $\text{rank}(A) + \text{rank}(B) \leq 4$.

Problem 4. Prove or find a counterexample:

a) $\mathrm{C}(A^T A) = \mathrm{C}(A)$.

   False.

   Counterexample, consider the case of where A is 1×4 matrix. Then $dimC(A^T A) = 1$, which does not match to the dimension of $C(A) = 4$, and thus $\mathrm{C}(A^T A) \neq \mathrm{C}(A)$.

b) $\mathrm{C}(A^T A) = \mathrm{C}(A^T)$.

   True.

   Suppose A^T is a $m \times n$ matrix. Firstly, $C(A^T A) \in C(A^T)$ because any column in $C(A^T A)$ is a linear combination of columns in $C(A^T)$, and so every vector in $C(A^T A)$ is in the vector space of $C(A^T)$.

   We then go on with the examination of the dimension applying the Fundamental Theorem of Linear Algebra:

$$dimC(A^T A) + dimN(A^T A) = n$$

   Then, also notice:

$$dimC(A^T) + dimN(A) = n$$

because they are orthogonal complement, the column space of A transposes the null space of A.

$$dimC(A) + dimN(A) = n$$

As a result,

$$dimC(A^T A) = C(A^T)$$

c) N($A^T A$) = N(A^T).

False.

The most straightforward way is to compare the dimension similar to the process we did in part a). Suppose A is a 1×4 matrix, and $A^T a4 \times 1$ matrix. The null space of A is zero unless the column space is identical zero. On the other hand, $N(A^T A)$ is 4×4, with rank 1, the dimension of $N(A^T A)$ is at least 3 which does not match up the dimension of $N(A)$.

We can also express $A^T Ax = 0$ for some vector x. Similarly, $A^T y = 0$ for some vector y by the definition of being in the null space.

The null space of $A^T A$ contains vectors that also belong to the null space of A^T since any vector in the null space of A^T will also be in the null space of $A^T A$ due to the multiplication by A, but it's not necessarily the case that they're equal.

For them to be equal, it would mean that every vector x such that $A^T Ax = 0$ would also satisfy $A^T x = 0$. and this is only going to hold if A has full column rank.

Problem 5. If P is the plane of vectors in $\mathbb{R}^4$ satisfying $x_1 + x_2 + x_3 + x_4 = 0$, write a basis for $P^\perp$. Construct a matrix that has P as its null space.

The equation $x_1 + x_2 + x_3 + x_4 = 0$ can be rewritten in the matrix form

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0.$$

Thus P is the null space of the 1×4 matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}.$$

This implies that $P^\perp$ is the row space of A. Obviously, a basis of $P^\perp$ is given by the vector

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Problem 6. Suppose that A is a symmetric matrix, i.e., $A = A^T$.

a) Show that the column space of A is perpendicular to the null space of A.

   In which case $C(A) = \text{row space}(A^\top) = \text{row space}(A)$. Now, By the definition of the null space, $N(A)$ contains all vectors v such that $Av = 0$. Letting $a_1, \ldots, a_n$ be the rows (and columns) of A, matrix multiplication tells us that $a_i \cdot v = 0$ for each $i = 1, \ldots, n$. Thus any vector $v \in N(A)$ is orthogonal to $C(A)$. It follows that $C(A) \perp N(A)$.

b) If $A\tilde{x} = 3\tilde{x}$ and $A\tilde{z} = 5\tilde{z}$, show that $\tilde{x}^T\tilde{z} = 0$.

   Let's start by considering:

   Because $A = A^T$
   $$Ax = 3x \Rightarrow A^T x = 3x$$
   Similarly,
   $$Az = 5z \Rightarrow A^T z = 5z$$
   Now, take the dot product of the equations we have:
   $$x^T Az = x^T A^T z = (Ax)^T z = (3x)^T z = 3x^T z$$
   $$x^T Az = 5x^T z$$
   So we have
   $$3x^T z = 5x^T z$$
   This implies that
   $$x^T z = 0$$

Problem 7. If a matrix P satisfies $P^2 = P$, it is called a projection. If in addition it satisfies $P^T = P$, then it is called an orthogonal projection.

a) Show that if P is an orthogonal projection, then $(I - P)$ is also an orthogonal projection.

   Now, let's consider $(I - P)^2$:
   $$\begin{aligned}
   (I-P)^2 &= (I-P)(I-P) \\
   &= I^2 - IP - PI + P^2 \\
   &= I - 2P + P^2 \\
   &= I - 2P + P \quad (\text{since } P^2 = P \text{ for an orthogonal projection}) \\
   &= I - P
   \end{aligned}$$
   So, we have shown that $(I - P)^2 = (I - P)$, which satisfies the property of an orthogonal projection.

Next, let's consider $(I-P)^T$:

$$\begin{aligned}(I-P)^T &= (I-P)^T \\ &= I^T - P^T \\ &= I - P \quad \text{(since } P^T = P \text{ for an orthogonal projection)}\end{aligned}$$

So, we have shown that $(I-P)^T = (I-P)$, which also satisfies the property of an orthogonal projection.

Hence, $(I-P)$ is an orthogonal projection if P is an orthogonal projection.

b) Show that if P is an orthogonal projection, then for any vectors $\vec{u}$ and $\vec{v}$, the vectors $P\vec{u}$ and $(I-P)\vec{v}$ are orthogonal, i.e., $(P\vec{u})^T(I-P)\vec{v} = 0$.

To show that Pu and $(I-P)v$ are orthogonal for any vectors u and v, we need to show that their dot product is zero.

Consider the dot product of Pu and $(I-P)v$:

$$\begin{aligned}(Pu)^T(I-P)v &= u^T P^T (I-P)v \quad \text{(since } (AB)^T = B^T A^T\text{)} \\ &= u^T P^T v - u^T P^T P v \\ &= u^T P v - u^T P^2 v \quad \text{(since } P^T = P \text{ for an orthogonal projection)} \\ &= u^T P v - u^T P v \\ &= 0\end{aligned}$$

So, we have shown that the dot product of Pu and $(I-P)v$ is zero, implying that they are orthogonal.

c) Show that for any vector $\vec{v}$, $||P\vec{v}|| \leq ||\vec{v}||$.

For any vector v, we know that P is an orthogonal projection, so $\mathbf{Pv}$ is the projection of v onto the subspace spanned by the columns of P. As a result, we can express v as follows:

$$v = Pv + (I-P)v$$

Because this is an orthogonal projection, we know that the Pythagorean Theorem can be applied here:

$$||\vec{v}||^2 = ||P\vec{v}||^2 + ||(I-P)\vec{v}||^2$$

$$||\vec{v}||^2 \geq ||P\vec{v}||^2 \Rightarrow ||P\vec{v}|| \leq ||\vec{v}||$$

as desired.

Problem 8. Project the vector $b = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$ onto the column space of the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$. Find the vector $e = b - b^*$ where b^* is the projection of b onto the column space.

To project the vector b onto the column space of matrix A, we need to find the projection matrix P and then compute the projection $b^* = Pb$.

The projection matrix onto the column space of A is given by $P = A(A^TA)^{-1}A^T$.

First, let's compute A^TA:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}$$

Now, let's compute $(A^TA)^{-1}$:

$$(A^TA)^{-1} = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3/2 & -1 \\ -1 & 1 \end{bmatrix}$$

Then, we have $P = A(A^TA)^{-1}A^T$:

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3/2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, let's compute $b^* = Pb$:

$$b^* = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$$

Finally, we compute the error vector $e = b - b^*$:

$$e = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So, the projection of b onto the column space of A is $b^* = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$ and the vector $e = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.