Math 3406 HW3

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Problem 1. Show that the inverse of a lower triangular $m \times m$ matrix is lower triangular.

Proof. Proof by induction.

Suppose there is the lower triangular matrix A, and its inverse matrix B.

Base case ($n = 2$): Consider a 2×2 matrix. Note that $AB = I$. Therefore, $a_{11}b_{12} + a_{12}b_{22} = 0$. However, we know that $a_{12} = 0$ because the matrix is lower triangular. Therefore, $a_{11}b_{12} = 0$. Note that a_{11} cannot equal zero as a leading coefficient; therefore, b_{12} must be zero for the statement $AB = I$ to be generally true. Therefore, B must also be lower triangular.

Now, consider an $n \times n$ matrix A and its inverse B. Any column j except $j = n$, we have $0 = \sum_{i=1}^{n} a_{ij}b_{in}$. However, a_{nn} should be the only non-zero entry of A in this sum, because otherwise, matrix A will have linearly dependent vectors which will make the dimension less than m. Thus $0 = a_{nn}b_{in}$. Since a_{nn} is not zero, it must be that b_{in} is always zero for $i \neq n$.

Since A and B are inverses, $AB = BA = I$. Therefore, considering $BA = I$, we can similarly multiply row i of B with column 1 of A. For all $i \neq 1$, we know that $0 = \sum_{i=1}^{n} a_{ij}b_{in} = a_{11}b_{1j}$. Since a_{11} is not zero, it must be that $b_{1j} = 0$.

By induction, the inverse of A without the first column and row is lower triangular. We have shown that the first column and last rows satisfy the conditions for lower triangular. Therefore, the inverse of an $n \times n$ lower triangular matrix is lower triangular. □

Proof. Let $A = [x, \ldots, x_n]$, where each x_k is an element in column space of the matrix A, and we know that $A^{-1}A = I = [e_1 \ldots e_n]$. Now we consider process of row reduction from matrix A to I,

$$A^{-1}A = E_n E_{n-1}, \ldots, E_1 A = I$$

To reduce matrix A to identity, observe that we only need to care about the element under the diagonal matrix. since e_k has only 0s above the k-th row and A is lower triangular, and $A^{-1}x_k = e_k$, then x_k has only 0s above the k-th row. This is true for all $1 \leq k \leq n$, so since $A = [x_1 \ldots x_n]$, then A^{-1} is lower triangular, too. □

Problem 2. If the null space of a matrix A, $N(A)$, is given by all multiples of $\begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, what is R and what is its rank?

Suppose we have matrix A in row reduced echelon form. We observe that in the null space, there are two entries x_2 and x_4 equal to 1, we know there is a free variable. Let x_4 be the free variable:

$$\begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From above, we can manipulate to make the multiplication work:

$$\begin{aligned} x_1 &= -2x_4 \\ x_2 &= -x_4 \\ x_3 &= -x_4 \end{aligned}$$

and so, we get:

$$R = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Given that the null space of A is spanned by $\vec{x}$, by definition, $A\vec{x} = \vec{0}$. There must 4 columns in A for matrix multiplication to be reasonable. So there is maximal column rank of 4, and there is only 1 basis in null space which span A, so $N(A) = 0$. By the fundamental theorem of Linear Algebra(rank-nullity theorem), the rank r should be:

$$r = 4 - 1 = 3$$

Problem 3. Suppose you know that the 3×4 matrix A has the vector $\vec{s} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$ as the only special solution to $A\vec{x} = \vec{0}$.

a) What is the rank of A and the complete solution of $A\vec{x} = \vec{0}$?

The matrix A has only one free variable x_a and one free column, the 3rd column. The rank of A is $\text{rank}(A) =$ number of pivot columns $= 4-1 = 3$.

The complete solution to $Ax = \mathbf{0}$ is $\begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix} x_a = c \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$, where c is a scalar.

b) What is the exact row reduced echelon form R_0 of A?

Since $\mathbf{s} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$, the exact reduced row echelon form R of A is given by

$$R = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

c) How do you know that $A\vec{x} = \vec{b}$ can be solved for all $\vec{b}$?

The matrix has full row rank $r = 3$, so there always is at least one solution (and since there is a free variable, so $Ax = \vec{b}$ can be solved for all $\vec{b} \in \mathbb{R}^3$

Problem 4. Decide whether the following list of vectors are linearly dependent or independent:

$$a) \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$[A|\vec{0}] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 3 & 1 & 2 & 0 \\ 2 & 3 & 1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The only solution is the trivial solution $a_1 = a_2 = a_3 = 0$, so the vectors are linearly independent.

$$b) \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

$$[B|\vec{0}] = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ -3 & 1 & 2 & 0 \\ 2 & -3 & 1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since there is a free variable at a_3, so there are infinite many solutions, so these vectors are linearly dependent.

Problem 5. If $\vec{w}_1$, $\vec{w}_2$, $\vec{w}_3$ are independent vectors, show that the vectors $\vec{v}_1 = \vec{w}_1 + \vec{w}_2$, $\vec{v}_2 = \vec{w}_2 + \vec{w}_3$, and $\vec{v}_3 = \vec{w}_3 + \vec{w}_1$ are linearly independent.

To show that the vectors $\vec{v}_1 = \vec{w}_1 + \vec{w}_2$, $\vec{v}_2 = \vec{w}_2 + \vec{w}_3$, and $\vec{v}_3 = \vec{w}_3 + \vec{w}_1$ are linearly independent, we can use the definition of linear independence. Let's assume that there exist scalars c_1, c_2, and c_3 such that:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

Now, substitute the expressions for $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$:

$$c_1(\vec{w}_1+\vec{w}_2)+c_2(\vec{w}_2+\vec{w}_3)+c_3(\vec{w}_3+\vec{w}_1)=\vec{0}$$

Distribute the scalars:

$$c_1\vec{w}_1+c_1\vec{w}_2+c_2\vec{w}_2+c_2\vec{w}_3+c_3\vec{w}_3+c_3\vec{w}_1=\vec{0}$$

Now, group the terms:

$$(c_1+c_3)\vec{w}_1+(c_1+c_2)\vec{w}_2+(c_2+c_3)\vec{w}_3=\vec{0}$$

Since $\vec{w}_1$, $\vec{w}_2$, and $\vec{w}_3$ are independent, the coefficients must be zero:

$$c_1+c_3=c_1+c_2=c_2+c_3=0$$

Solving the above equality with matrix, we get:

$$\left[\begin{array}{ccc|c}1&0&1&0\\1&1&0&0\\0&1&1&0\end{array}\right]=\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right]$$

These equations imply that $c_1=c_2=c_3=0$, which means that the vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are linearly independent.

Problem 6. Fill out the matrices so that they have rank 1:

$$A=\begin{bmatrix}a&b&c\\d&&\\g&&\end{bmatrix},B=\begin{bmatrix}&9&\\1&&\\2&6&3\end{bmatrix},C=\begin{bmatrix}a&b\\c&\end{bmatrix}$$

$$A=\begin{bmatrix}a&b&c\\d&\frac{bd}{a}&\frac{cd}{a}\\g&\frac{bg}{a}&\frac{cg}{a}\end{bmatrix}\xrightarrow[r_2\to r_2-\frac{d}{a}r_1]{r_3\to r_3-\frac{g}{a}r_1}\begin{bmatrix}a&b&c\\0&0&0\\0&0&0\end{bmatrix}$$

$$B=\begin{bmatrix}3&9&\frac{9}{2}\\1&3&\frac{3}{2}\\2&6&3\end{bmatrix}\xrightarrow[r_3\to r_3-\frac{2}{3}r_1]{r_2\to r_2-\frac{1}{3}r_1}\begin{bmatrix}1&3&\frac{3}{2}\\0&0&0\\0&0&0\end{bmatrix}$$

$$C=\begin{bmatrix}a&b\\c&\frac{bc}{a}\end{bmatrix}\xrightarrow{r_2\to r_2-cr_1}\begin{bmatrix}1&\frac{b}{a}\\0&0\end{bmatrix}$$

As shown on the above, rank is equal to the number of pivots in R.

Problem 7. Show that rank$(AB) \leq$ rank(A) and that rank$(AB) \leq$ rank(B). (Hint for the second statement: Show that $N(B) \subset N(AB)$ and hence the number of free variables of AB is greater or equal to the number of free variables of B.)

Proof.
1. rank$(AB) \leq$ rank(A)

We first show that $C(AB) \subseteq C(A)$. Let any $x \in C(AB)$, we can choose any vector y such that $x = (AB)y = A(By) \in C(A)$. Then, because $C(AB) \subseteq C(A)$, dim $C(AB) \leq$ dim $C(A)$. As a result, rank$(AB) \leq$ rank(A).

2. rank$(AB) \leq$ rank(B)

We first show that $N(B) \subseteq N(AB)$. Let some $x \in N(B)$, we have $(AB)x = A(Bx) = A\mathbf{0} = \mathbf{0}$, such that $x \in N(AB)$. Then, because $N(B) \subseteq N(AB)$, the number of free variables of AB is greater or equal to the number of free variables of B. As a result, dim $N(B) \leq$ dim $N(AB)$.Thus, by the Fundamental Theorem of Linear Algebra, rank$(AB) = n -$ dim N$(AB) \leq n -$ dim N$(B) =$ rank(B).

Alternatively, a more direct approach to rank$(AB) \leq$ rank(B). Suppose we take transpose of AB, because $(AB)^T = B^T A^T$, we can consider rows of (AB) as a linear combination of rows of B, we claim that $C((AB)^T) \in C(B^T)$, and thus dim C$((AB)^T) \leq$ dim C(B^T). Lastly, we know that dim $C(A) =$ dim $C(A)^T$. Therefore, we conclude that rank$(AB) \leq$ rank(B).

In fact, with these two inequality holds, we can conclude that Rank$(AB) \leq$ min(Rank(A), Rank(B))

□

Problem 8. Suppose that A, B are $n \times n$ matrices and that $AB = I$, i.e., B is the right inverse of A. We want to prove that B is also the left inverse of A, i.e., $BA = I$. (Hint: What is the rank of A?)

Following the hint, existence of a right inverse implies the existence of a left inverse when dealing with square matrices. Additionally, for a square matrix A, if it has a right inverse, then its rank must be full. (i.e., rank(A) = n).

Proof. Let's multiply the equation $AB = I$ by A from the right side. We get

$$(AB)A = A \Rightarrow A(BA) = A \Rightarrow A(BA - I) = 0$$

Now, we can write $(BA - I)$ as vectors in columns: $v_1, v_2, \ldots, v_n$, which allow us to express $Av_1 = 0, Av_2 = 0, \ldots, Av_n = 0$. Since rank(A) = n, there are n column vectors of A that span $\mathbb{R}^n$, and so they must be independent. Then $Av = 0$ implies $v = 0$. Hence, $v_1 = v_2 = \ldots = v_n = 0$, so $BA - I$ is the zero matrix, and finally $BA = I$. □