Math 3406 HW2

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Problem 1: Find a 3×3 permutation matrix $P \neq I_3$ with $P^3 = I_3$.

From problem, we can infer that $P^2 \neq I$, because

$$P^2 = P^3 \implies (P^2)I_3 = (P^2)P \implies I_3 = P,$$

which contradicts with the condition provided that $P \neq I_3$. Then

$$P^2 \neq I \implies P^T P^2 \neq P^T I \implies P \neq P^T.$$

As a result, P should be a non-symmetric permutation matrix, so besides the identity matrix, the left non-symmetric matrix are:

$$P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, P_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Problem 2: Show that any $n \times n$ matrix M can be written as $M = S + A$ where $S^T = S$ and $A^T = -A$.

Let M be an $n \times n$ matrix. We can decompose M into a symmetric matrix S and an anti-symmetric matrix A:

$$M = \frac{1}{2}(M + M^T) + \frac{1}{2}(M - M^T)$$

Let $S = \frac{1}{2}(M + M^T)$ and $A = \frac{1}{2}(M - M^T)$.

$$S^T = \left(\frac{1}{2}(M + M^T)\right)^T = \frac{1}{2}(M^T + M) = \frac{1}{2}(M + M^T) = S$$

$$A^T = \left(\frac{1}{2}(M - M^T)\right)^T = \frac{1}{2}(-M^T + M) = -\frac{1}{2}(M + M^T) = -A$$

Therefore, any $n \times n$ matrix M can be written as the sum of S and A, where $S^T = S$ and $A^T = -A$.

Problem 3: Find a condition on b_1, b_2, b_3 such that the systems below are solvable:

$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

a) For the first system, we combine matrix A and $\vec{B}$ to get augmented matrix:

$$[A|\vec{B}] = \left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 2 & 8 & 4 & b_2 \\ -1 & -4 & -2 & b_3 \end{array}\right]$$

Now, we perform row operations to transform matrix to its row reduced echelon form, and we get

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 + b_1 \end{array}\right]$$

We found that on the second and third row, the leading coefficient are 0 and there is not pivot. As a result, in order for this linear system to have solution, we must have:

$$b_2 = 2b_1, b_3 = -b_1$$

b) Similar to part(a). For the second system, we combine matrix C and $\vec{D}$ to get augmented matrix:

$$[C|\vec{D}] = \left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 2 & 9 & b_2 \\ -1 & -4 & b_3 \end{array}\right]$$

Now, we perform row operations to transform matrix to its row reduced echelon form, and we get

$$\left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & b_3 + b_1 \end{array}\right]$$

We found that on the third row, the leading coefficient are 0 and there is not pivot. As a result, in order for this linear system to have solution, we must have:

$$b_3 = -b_1$$

Overall, one could argue that simply check whether two matrices' rank are equal to answer if there is a solution to the system.

Problem 4: Assume that V and W are subspaces of $\mathbb{R}^n$. Show that their intersection $V \cap W$ is a subspace of $\mathbb{R}^n$. Just check the definition of a subspace. By definition of a subspace,

1. for $v, w \in V \implies v + w \in V$
2. if λ is a scalar, and $v \in V$, then $\lambda v \in V$
3. $0 \in V$

Now suppose $v, w \in V \cap W$, then $v \in V$, $v \in W$, and $w \in V$, $w \in W$. Because $v \in V$ and $w \in V$, we can infer that $v + w \in V$. Similarly, because $v \in W$ and $w \in W$, we can infer that $v + w \in W$. That is to say, vector $v + w$ is both in V and W, so $v + w \in V \cap W$.

Then, let a new vector $x \in V \cap W$, and let a constant $\lambda \in \mathbb{R}$. As stated previously, $x \in V$, and $x \in W$. By definition of subspace, $\lambda x \in V$ and $\lambda x \in W$. It follows that $\lambda x \in V \cap W$.

Lastly, As V and W are subspaces of $\mathbb{R}^n$, the zero vector $\mathbf{0}$ is in both V and W. Hence, the zero vector $\mathbf{0} \in V \cap W$.

By checking 3 conditions, we verify that $V \cap W$ is a subspace of $\mathbb{R}^n$.

Problem 5: Construct a matrix whose column space contains the vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$ and whose null space contains $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

To construct a matrix whose column space contains a given vector , set that vector a column of the matrix. Hence, let

$$A = \begin{bmatrix} 1 & 0 & a_1 \\ -1 & 3 & a_2 \\ 0 & 1 & a_3 \end{bmatrix}.$$

Then the column space of A already contains two vectors, and we just need to find a_1, a_2, a_3 such that $(1, 1, 2)$ is in the nullspace. That is to say, we choose a_1, a_2, a_3 such that A $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = 0$. So we perform matrix multiplication:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a_1 \\ -1 & 3 & a_2 \\ 0 & 1 & a_3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 + 2a_1 \\ 2 + 2a_2 \\ 1 + 2a_3 \end{bmatrix}.$$

Hence, we can pick $a_1 = -1/2$, $a_2 = -1$, and $a_3 = -1/2$, so the matrix is:

$$A = \begin{bmatrix} 1 & 0 & -1/2 \\ 1 & 3 & -1 \\ 5 & 1 & -1/2 \end{bmatrix}$$

Problem 6: For the two matrices below, find their row reduced echelon form.

$$\begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}$$

Circle the pivots, find the special solution for each free variable, and describe every solution of the system $A\vec{x} = \vec{0}$. Write A in the form $A = CR$.

a) For the first matrix,

$$\begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \xrightarrow{r_2 \to r_2 - r_1} \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \xrightarrow[r_1 \to r_1 - 2r_2]{r_3 \to r_3 - r_2} \begin{bmatrix} \textcircled{1} & 2 & 0 & 0 & 0 \\ 0 & 0 & \textcircled{1} & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The circled entries are the pivots, and we have one pivot in the first column and one in the third column. The free variables are x_2,x_4 and x_5.

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = -2x_2$$
$$x_3 = -2x_4 - 3x_5$$

The solution set for the system $A\vec{x} = \vec{0}$ is given by:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ -2x_4 - 3x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

The special solution for each free variable:

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

Write A in the form $A = CR$.

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$$

b) Now, for the second matrix B:

$$\begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix} \xrightarrow[r_3 \to r_3 - 2r_2]{r_1 \to \frac{1}{2} r_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[r_2 \to \frac{1}{4} r_2]{r_1 \to r_1 - 2r_2} \begin{bmatrix} \textcircled{1} & 0 & -1 \\ 0 & \textcircled{1} & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The circled entries are the pivots, and we have one pivot in the first column and one in the second column. The free variable is x_3.

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = x_3$$
$$x_2 = -x_3$$

The solution set for the system $B\vec{x} = \vec{0}$ is given by:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

The special solution for the free variable:

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Expressing the original matrices A and B in the form CR:

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 0 & 4 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

Problem 7: Show by example that the following statements are generally false:

a) The matrices A and A^T have the same null space.

Consider the matrix A:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

The null space of A is the set of all vectors $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$. In this case, the null space of A is $\text{null}(A) = \text{span}\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\}$.

Now, let's find the null space of A^T:

$$A^T = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$$

The null space of A^T is $\text{null}(A^T) = \text{span}\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$.

Since null(A) and null(A^T) are not the same, statement (a) is generally false.

b) A and A^T have the same free variables. Consider the matrix A:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

The free variable in this case is the second variable. However, in the transpose of A^T, the free variable would be the first variable. Thus, statement (b) is generally false.

c) If R is the row reduced echelon form of A, then R^T is the row reduced echelon form of A^T.

Consider the matrix A:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

The row reduced echelon form of A is:

$$R = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Now, the transpose of R^T is:

$$R^T = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$$

This is not the row reduced echelon form of A^T because the second column in R^T is not a unit column vector. Therefore, statement (c) is generally false.

Problem 8:

a) Construct a 2×2 matrix whose column space equals its null space.

If the nullspace is equal to the column space,then the matrix must have $m = n$. By the rank-nullity theorem, n must also be an even number.

Consider the matrix A where $n = 2$:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(a) Column Space: The column space of A is the span of its column vectors. In this case, the second column is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(b) Null Space: The null space of A is the set of all vectors that satisfy $A\vec{x} = \vec{0}$. For the matrix A, if we take $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $A\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $\vec{x}$ is in the null space.

b) Find all five 2×2 matrices that have entries 0 and 1 and that are in row reduced echelon form.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$