Math 3406 HW1

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Problem 1: A system of linear equations cannot have precisely two solutions. Let $\vec{x}_1$ and $\vec{x}_2$ be two solutions of $A\vec{x} = \vec{b}$. Find other solutions.

solution: Given that $\vec{x}_1$ and $\vec{x}_2$ are two solutions of $A\vec{x} = \vec{b}$, we can express the system to be $A\vec{x}_1 = A\vec{x}_2 = \vec{b}$. Then for any $\lambda \in (0, 1), \lambda\vec{x}_1 + (1 - \lambda)\vec{x}_2$ will be the solutions for this linear equation, reason shown as follows:

$$A(\lambda\vec{x}_1 + (1 - \lambda)\vec{x}_2) = \lambda A\vec{x}_1 + (1 - \lambda)\vec{x}_2 = \lambda\vec{b} + (1 - \lambda)\vec{b} = \vec{b}$$

and we know that $\lambda\vec{x}_1$ is different from $\vec{x}_1$, same for $\vec{x}_2$, and we can pick $\lambda = 1/2$ as an example:

$$A(\frac{1}{2}\vec{x}_1 + (1 - \frac{1}{2})\vec{x}_2) = \frac{1}{2}A\vec{x}_1 + \frac{1}{2}A\vec{x}_2 = \frac{1}{2}\vec{b} + \frac{1}{2}\vec{b} = \vec{b}$$

Problem 2: Which numbers a would leave the matrix A with two independent columns?

solution: For a 3×3 matrix to have two independent columns, one of the columns must be linearly a linear combination of the other two linearly independent columns .

$$A_1 = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 9 \\ 5 & 0 & a \end{bmatrix}$$

For this matrix, the third column $\begin{bmatrix} 2 \\ 9 \\ a \end{bmatrix}$ since the first two column are not scalar multiple or linear independent of each other, the third column should be a linear combination of them, such linear combination can be $2\vec{v}_1 + 3\vec{v}_2$, so a should be equal to 10.

$$A_2 = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & a \end{bmatrix}$$

again, similar to the first matrix, the first two column are not linear independent of each other, the third column should be a linear combination of them, such linear combination can be $-\vec{v}_1 + 2\vec{v}_2$. We found that the if we take linear combination, the third row's left with 0, so a should be equal to 0.

$$A_3 = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 0 & 0 & a \end{bmatrix}$$

For this matrix, the first two columns are identical, so they are not linearly independent. as not for last column to be linearly independent with the first two columns, pick any $a \neq 0$ will satisfy the condition.

Problem 3: Using Schwarz's inequality prove the triangle inequality for vectors in $\mathbb{R}^n$

Proof. Let u and v be vectors in $\mathbb{R}^n$. Schwarz's Inequality states that for any vectors u and v in $\mathbb{R}^n$,

$$|u \cdot v| \leq \|u\|\|v\|$$

Now, we want to prove the Triangle Inequality applying Schwarz's Inequality:

$$\begin{aligned} ||u+v||^2 &= (u+v, u+v) \\ &= (u,u) + (v,v) + 2\text{Re}(u,v) \\ &\leq ||u||^2 + ||v||^2 + 2|(u,v)| \\ &\leq ||u||^2 + ||v||^2 + 2||u|||v|| \\ &= (||u|| + ||v||)^2 \end{aligned}$$

Taking square roots of both sides of the inequality above gives the triangle inequality Therefore, $||u+v|| \leq ||u|| + ||v||$ □

Problem 4: Consider the plane P spanned by the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, and consider the vector $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find the point on the plane that is closest to the tip of the vector $\vec{b}$.

Suppose we have a vector $\vec{b^*}$ on the plane that intersect with vector $\vec{b}$ on the plane, and another vector $\vec{n}$ orthogonal to the plane such that the vector addition holds: $\vec{b} - \vec{b^*} = \vec{n}$. Such vector $\vec{n}$ can be expressed as $A\vec{n} = 0$, where A is spanned by vectors $\vec{v_1}$ and $\vec{v_2}$:

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = z \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Then, we can assume that there exists a scalar $t \in R$ such that

$$(\vec{b} - \vec{b^*}) = t\vec{n}$$

$t\vec{n}$ is the vector that is perpendicular to the plane and passes the tip of the vector $\vec{b}$. Then, by definition of orthogonality, we know that the dot product of $\vec{b^*} \cdot \vec{n} = 0$. Therefore, we have:

$$(\vec{b} - \vec{b^*}) \cdot \vec{n} = t\vec{n} \cdot \vec{n}$$

$$\vec{b} \cdot \vec{n} = t\vec{n} \cdot \vec{n}$$

$$\vec{b} \cdot \vec{n} = t||n||^2$$

$$t = \frac{\vec{b} \cdot \vec{n}}{||n||^2}$$

$$t\vec{n} = \frac{\vec{b} \cdot \vec{n}}{||n||^2}\vec{n}$$

$$t\vec{n} = \frac{1+2+3}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

so here we find the $t\vec{n}$ which is the the perpendicular component, we left to find $\vec{b^*}$ on the plane, which is:

$$\vec{b^*} = \vec{b} - t\vec{n} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Another way to approach this problem involving projection. The projection works the same way as the approach listed above, but it can be applied in much more occasions (higher dimensions). Suppose we have the same definition of variables, that is $\vec{b^*}$ as a vector on the plane A, and the vector $\vec{n}$ that is orthogonal to the plane. Similarly, We can write as $\vec{b} - \vec{b^*} \perp C(A)$, which can also easily seen that $\vec{b} - \vec{b^*} \in N(A^T)$. Furthermore, since $\vec{b^*}$ as a vector on the plane, $\vec{b^*} \in C(A)$, and so we can have some vector $\vec{x}$ such that $b^* = Ax$. Then, given $\vec{b} - \vec{b^*} \in N(A^T)$, we have:

$$0 = A^T(b - b^*) = A^Tb - A^TA\vec{x} \rightarrow A^TA\vec{x} = A^Tb$$

In fact, this is called the **Normal Equations**. Then:

$$x = (A^TA)^-1A^Tb$$

$$b^* = Ax = A(A^TA)^{-1}A^Tb$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

As desired.

Problem 5: Completing a 2 × 2 matrix to have

rank one:

$$\begin{bmatrix} 3 & 6 \\ 5 & 10 \end{bmatrix}$$

To make it a rank-one matrix, we need the second column to be a scalar multiple of the first column, so that there is only 1 linearly independent column. Choose $\lambda = 2$

orthogonal columns:

$$\begin{bmatrix} 5 & 8 \\ 8 & -5 \end{bmatrix}$$

For columns to be orthogonal, the dot product of the columns should be zero. Choose -5

rank 2:

$$\begin{bmatrix} 2 & 1 \\ 3 & 6 \end{bmatrix}$$

As long as two columns are linearly independent, the rank C(A) will be 2.

$A^2 = I$:

$$\begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}^2 = \begin{bmatrix} a^2 + bc & ac + bd \\ ab + cd & bc + d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

To get the identity matrix, either $a = -d$, so $a^2 + bc = 1$ solve for the equation and we get that the b should be equal to -2

Problem 6: Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 5 & 4 & 1 \\ -2 & 2 & 0 \end{bmatrix}$$

Find the three elimination matrices E_{21}, E_{31}, E_{32} such that $E_{32}E_{31}E_{21}A = U$ where U is in upper triangular form.

$$E_{21}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y - 5x \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} E_{21}A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ -2 & 2 & 0 \end{bmatrix}$$

$E_{31} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z+2x \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} E_{31}E_{21}A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 4 & 0 \end{bmatrix}$
$E_{32} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ \frac{1}{4}Z \\ y+\frac{1}{4}Z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{4} \\ 0 & 1 & \frac{1}{4} \end{bmatrix} U = E_{32}E_{31}E_{21}A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
As shown, U is in upper triangular form.

Problem 7: Suppose A is an invertible matrix, and you exchange row 1 and row 2 to get a matrix B. Is the matrix B invertible? If yes, how would you find B^{-1} in terms of A^{-1}?

Since B is obtained from A by exchanging the first two rows, we know

$$B = E_{21}A$$

with

$$E_{21} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

So, $BA^{-1}E_{21} = E_{21}AA^{-1}E_{21} = E_{21}E_{21} = I$ and hence $B^{-1} = A^{-1}E_{21}$. In other words, B^{-1} is obtained from A^{-1} by exchanging the first two columns.

Problem 8: Suppose a 4×4 matrix A contains in every row the numbers $0, 1, 2, -3$ in some order. Can it be invertible?

No, it can't be invertible. Since every row the number is in different order, it is guaranteed that every element from same column adds up to 0, which means the null space is not just 0. For example, multiply matrix by the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ will result in 0. From class, we learnt that matrix A is invertible if and only if the null space N(A) = 0, which has shown not the case, so the matrix A is not invertible.